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NOTE ON GEOMETRICAL PRODUCTS

BY C. L. E. MOORE AND H. B. PHILLIPS

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Geometrical products may be divided into two general classes: those definable in a space of any number of dimensions and those definable only in a space of a definite number of dimensions. To the first class belong the progressive (outer) product of Grassmann and the inner products of Grassmann, Gibbs[1] and Lewis.[2] To the second class belong the regressive product of Grassmann and the cross product of Gibbs (as defined in the Gibbs-Wilson Vector Analysis) and the quaternion multiplications. In this paper we show that there is a series of geometrical products independent of dimensions that may be in a sense considered intermediate between the progressive product and the inner product.

In terms of the units[3] the progressive product is expressed by identities of the form $k_{12}k_{34} = k_{1234}$, the product containing all the units multiplied together. If there are any common units as in $k_{12}k_{14} = 0$, the product is zero. The inner product of Lewis is expressed by identities of the form $k_3.k_{123} = k_{12}$, the common units being cancelled (with certain conventions as to algebraic sign). Unless one factor is entirely contained in the other the result is zero.

If π_1 and π_2 are products of units whose first m factors are the same and in the same order we define as the product $[\pi_1\pi_2]_m$ of index m the result of cancelling those m common units and taking the outer product of the remaining units in the order written. Thus

$$[k_{12}k_{13}]_1 = k_{23}, \; [k_{123}k_{124}]_2 = k_{34}.$$

If π_1 and π_2 have m units in common but these are not the first m units, π_1 and π_2 are rewritten (with change of sign if necessary as required by the outer multiplication) in such a way that the first m factors are the same, and in the same order, the product is then formed as above, negative signs introduced being included.[4] Thus

$$[k_{12}k_{23}]_1 = -k_{13}, \; [k_{234}k_{134}]_2 = k_{21} = -k_{12}.$$

If there are either more or less than m units in common, the product is zero. Thus

$$[k_{123}k_{124}]_1 = k[_{123}k_{124}]_3 = 0.$$

The inner product is obtained when the index is equal to the dimension of the lowest factor. Thus

$$[k_1k_{12}]_1 = k_1.k_{12} = -k_2.$$

The outer product occurs when the index is zero. Thus $[k_{12}k_{34}]_0 = k_{1234}$.

To obtain the product of M and N, two homogeneous linear functions of units, they are multiplied distributively, the product of units being found as above. We shall now show that the product of two such quantities is invariant, i. e., that it is independent of the system of units in terms of which it is expressed. We do this by showing that $[MN]_m$ can be expressed in terms of invariant operations on M, N and the identical transformation[5] (dyadic) I.

If $k_1, k_2, \ldots\ldots k_n$ are the unit one-vectors, the 1, 1 idemfactor is

$$I_1 = k_1k_1 + k_2k_2 + \ldots\ldots + k_nk_n.$$

Using the dot to express inner multiplication we then have

$$\begin{aligned} k_{12}.I_1 &= k_1k_2 - k_2k_1 = -I_1.k_{12}, \\ k_{123}.I_1 &= k_{23}k_1 - k_{13}k_2 + k_{12}k_3, \\ I_1.k_{123} &= k_1k_{23} - k_2k_{13} + k_3k_{12}, \end{aligned}$$

and similar expressions for products with units of higher order. Using these values we find

$$[k_{12}k_{13}]_1 = [(k_{12}.I_1).(I_1.k_{13})]_0,$$

where the subscript 0 indicates that the antecedent and consequent of each dyad in the brackets are multipled according to the (outer) multiplication of index zero and the sum of terms then taken. Similarly we find

$$[k_{123}k_{145}]_1 = [(k_{123}.I_1).(I_1.k_{145})]_0,$$

and generally

$$[\pi_1\pi_2]_1 = [(\pi_1.I_1).(I_1.\pi_2)]_0$$

where π_1 and π_2 are any product of units. Since the operations are all distributive we have then

$$[MN]_1 = [(M.I_1).(I_1.N)_0].$$

Since the dot multiplication, the outer (zero) multiplication and I_1 are invariant it follows that $[MN]_1$ is invariant.

For the product of index 2, we use the 2, 2 idemfactor $I_2 = \Sigma k_{ij}k_{ij}$ with $i<j$. We then find

$$[k_{123}k_{124}]_2 = [(k_{123}.I_2).(I_2.k_{124})]_0$$

and generally

$$[\pi_1\pi_2]_2 = [(\pi_1.I_2).(I_2.\pi_2)]_0,$$

or since the operations are distributive

$$[MN]_2 = [(M.I_2).(I_2.N)]_0$$

which shows the invariance as before.

In general the product of index m is

$$[MN]_m = [(M.I_m).(I_m.N)]_0$$

where

$$I_m = \Sigma k_{12\ldots\ldots m}\, k_{12\ldots\ldots m}.$$

Let M_p be a simple space of dimension p and N_q a simple space of dimension q.

Theorem I. In order that $[M_pN_q]_m = 0$ it is necessary and sufficient that

$$[M_p(R_m.N_q)]_0 = 0. \qquad (1)$$

where R_m is an arbitrary simple space of dimension m in M_p.

To prove this take the units $k_1, k_2, \ldots k_n$ so that $k_1, k_2, \ldots k_p$ lie in M_p and $R_m = k_{12\ldots m}$. Then

$$[M_pN_q]_m = \Sigma[(M_p.k_{12\ldots m})(k_{12\ldots m}.N_q)]. \qquad (2)$$

In the summation terms containing $k_{p+1}, k_{p+2}, \ldots k_n$ may be omitted since the product with M_p would be zero. The outer product of R_m and (2) is

$$[R_m(M_pN_q)_m]_0 = [R_m(M_p.k_{12\ldots m})(k_{12\ldots m}.N_q)]_0 = 0, \qquad (3)$$

all other terms vanishing because they contain repeated factors. Now

$$[R_m(M_p.k_{12\ldots m})]_0 = [k_{12\ldots m}(M_p.k_{12\ldots m})]_0 = M_p.$$

Hence (3) is equivalent to

$$[M_p(R_m.N_q)]_0 = 0$$

which was to be proved.

Theorem II. A necessary and sufficient condition that

$$[M_pN_q]_m = 0 \qquad (4)$$

is that either M_p and N_q intersect in a space of dimensions $\geqq m + 1$ or M_p contain a space of dimension $p - m + 1$ completely perpendicular to N_q and N_q contain a space of dimensions $q - m + 1$ completely perpendicular to M_p.

If (4) is satisfied, by (1), either $R_m.N_q = 0$ or the outer product of M_p and $R_m.N_q$ is zero. In the first case R_m contains a perpendicular to N_q and consequently M_p must contain a space perpendicular to N_q which is cut by every R_m. For this it is necessary and sufficient that M_p contain a space of dimensions $p - m + 1$ perpendicular to N_q. In the second case the intersection of M_p and N_q must contain a vector perpendicular to any R_m in M_p. The intersection must then be of such a dimension that it cuts all spaces in M_p of dimension $p - m$. It must then be of dimension $m + 1$.

Theorem III. If

$$[M_pN_q]_m = 0, \ [M_pN_q]_{m+1} \pm 0, \qquad (5)$$

M_p and N_q will intersect in a space of dimension $m + 1$ and so all the products of M_p and N_q of index equal to or less than m will be zero; and, if

$$[M_pN_q]_n = 0, \ [M_pN_q]_{n-1} \pm 0, \qquad (6)$$

M_p will contain a space of dimension $p - m + 1$ completely perpendicular to N_q and so all products of index equal to or greater than n will be zero.

In fact if relations (5) are satisfied, the vanishing of $[M_pN_q]_m$ must be due to M_p and N_q intersecting in a space of dimension $m + 1$; for if there were in M_p a space of dimension $p - m + 1$ perpendicular to N_q, the product $[M_pN_q]_{m+1}$ would also vanish. Hence, in this case all the products of index less than m will also vanish. If (6) is satisfied it must be due to M_p containing a space of dimension $p - m + 1$ completely perpendicular to N_q, for if M_p and N_q intersect in a space of order $m + 1$

the product $[M_pN_q]_{n-1}$ will also be zero, hence, all the products of index less than n will vanish.

It is clear from the theorem just proved that all the products $[M_pN_q]_r$ for which $m<r\underline{<}n$ will be different from zero and all the products for which $r\overline{<}m$ or $r\overline{>}n$ will be zero.

Theorem IV. If equations (5) are satisfied, $[M_pN_q]_{m+1}$ is the outer product of the parts of M_pN_q completely perpendicular to the intersection.

In this case M_p and N_q intersect in a space of $m+1$ dimensions. Let $k_1, k_2, \ldots k_{m+1}$ be unit vectors in that space. Then

$$M_p = k_1k_2\ldots k_{m+1}P$$

and

$$N = k_1k_2\ldots k_{m+1}Q$$

where P is the part of M_p perpendicular to $k_1k_1\ldots k_{m+1}$ and Q is the part of N_q completely perpendicular to $k_1k_1\ldots k_{m+1}$. Hence

$$[M_pN_q]_{m+1} = [PQ]_0$$

which proves the theorem.

Theorem V. If (6) is satisfied $[M_pN_q]_{n-1}$ is the outer product of the part of M_p perpendicular N_q and the part of N_q perpendicular to M_p.

Let $l_1, l_2, \ldots . l_{p-n+1}$ be the units in M_p perpendicular to N_q and $j_1, j_2, \ldots . j_{q-n+1}$ the units in N_q perpendicular to M_p. These units are then perpendicular to each other and

$$M_p = Pl_1l_2\ldots . l_{p-n+1}$$
$$N_q = Qj_1j_2\ldots . j_{q-n+1}$$

where P (of dimension $n-1$) is the part of M_p perpendicular to $l_1l_2\ldots l_{p-n+1}$ and Q is the part of N_q perpendicular to $j_1j_2\ldots . j_{q-n+1}$. Since the units $l_1l_2\ldots . l_{p-n+1}, j_1j_2\ldots . j_{q-n+1}$ are all distinct

$$[M_pN_q]_{n-1} = [PQ]_{n-1}[l_1l_2\ldots . l_{p-n+1}\, j_1j_2\ldots . j_{q-n+1}].$$

Since $[PQ]_{n+1}$ is a number this proves the theorem.

[1] Gibbs, Wilson, *Vector Analysis*.

[2] *Proc. Amer. Acad. Arts Sci.*, **46**, 1910 (165–181).

[3] The products here discussed occur as partial products in the multiplication of generalized quaternions. From this point of view they have been treated by Clifford (*Amer. J. Math.*, **I**), Joly (*Proc. Royal Irish Acad.*, **5**, (3)) McAuley (*Proc. R. Soc. Edinburgh*, **28**), Shaw (*Bull. Int. Assoc. Adv. Quaternions*, 1913, (24–27)), and others. In victor algebras of the Grassmann type, for example in the usual victor theory of relativity, they have, however, such a different appearance that we have presented them here from that point of view.

[4] In case the index is equal to the order of one of the factors the sign is taken as in the Lewis inner product.

[5] We used this as the definition of the product of index 1 (star product) in a paper "Rotations in Space of Even Dimensions" to appear in *Proc. Amer. Acad. Arts Sci.*